§11.4 Comparison Tests.

Suppose that $\sum a_n$, $\sum b_n$ are series with POSITIVE TERMS.

C.T. • Comparison Test: (i) If $\sum b_n$ is convergent and $\boxed{a_n \le b_n}$, then $\sum a_n$ is convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$, then $\sum a_n$ is convergent.

L.C.T. • Limit Comparison Test: If $\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = C > 0}$ (C is a positive finite number), then either BOTH series converge or BOTH diverge.

Motivation and Goal: Reduce the problem from UNKNOWN to KNOWN.

We have the conclusion for $\boxed{\text{p-series}}$ and $\boxed{\text{Geometric Series}}$. Consider some new series $\sum a_n$ which ARE SIMILAR to p-Series or G.S.. We choose $b_n = \frac{1}{n^p}$ or $a \cdot r^n$ to COMPARE with $\sum a_n$. Then the CONV/DIV of $\sum b_n$ will be passed on to $\sum a_n$.

$\boxed{\text{Key point: Choose } b_n \text{ carefully to draw the conclusion.}}$

Remark: C.T.: (i) ConV of LARGER one $\Rightarrow$ ConV of SMALLER one.

(ii) DIV of SMALLER $\Rightarrow$ DIV of LARGER

L.C.T.: Suppose $\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = 1}$. If $\sum b_n$ CONV, then $\sum a_n$ CONV.

If $\sum b_n$ DIV, then $\sum a_n$ DIV.

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e.g. 1 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ Hint: $\frac{1}{n^2+2}$ is similar to $\frac{1}{n^2}$, which is a p-series, $p=2$, ConV.

$$\underbrace{\frac{1}{n^2+2}}_{\substack{(a_n)\\ \text{smaller}}} < \underbrace{\frac{1}{n^2}}_{\substack{(b_n)\\ \text{larger}}}$$

Because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (p-Series, $p=2$), $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is also convergent

e.g. 2. $\displaystyle\sum_{n=2}^{\infty} \frac{3}{\sqrt{n}\cdot(\sqrt{n}-1)}$. $\displaystyle\frac{3}{\sqrt{n}\cdot(\sqrt{n}-1)} > \frac{3}{\sqrt{n}\cdot\sqrt{n}} = \frac{3}{n} \xleftarrow{b_n}$ p-Series, $p=1$

$\displaystyle\sum_{n=2}^{\infty} \frac{3}{n}$ is divergent, therefore $\displaystyle\sum_{n=2}^{\infty} \frac{3}{\sqrt{n}\cdot(\sqrt{n}-1)}$ is divergent.

☆ eg 3. $\sum_{n=1}^{\infty} \frac{1}{n+1}$. Remark: It is natural to relate this series to $\sum \frac{1}{n}$ and guess that $\sum \frac{1}{n+1}$ is divergent as $\sum \frac{1}{n}$.

$\boxed{\text{WRONG choice of } b_n: \ b_n = \frac{1}{n}}$

$a_n = \frac{1}{n+1}$, $b_n = \frac{1}{n}$, $n+1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n}$, i.e., $a_n < b_n$

We know that $\sum b_n = \sum \frac{1}{n}$ is divergent (p-Series, $p=1$). However, the comparison test is inconclusive for THIS CHOICE of b_n. (Neither (i) nor (ii) can be applied.)

$\boxed{\text{Correct Choice: } b_n = \frac{1}{2n}}$

$a_n = \frac{1}{n+1}$, $b_n = \frac{1}{2n}$, $n+1 \leq 2n \Rightarrow \frac{1}{n+1} \geq \frac{1}{2n}$, i.e., $a_n \geq b_n$.

$\sum a_n$ is larger, $\sum b_n$ is smaller. $\sum b_n = \sum \frac{1}{2n} = \frac{1}{2} \cdot \sum \frac{1}{n}$ is divergent. ($p=1$)

Therefore, according to Comparison Test (ii), $\sum b_n$ DIV implies $\sum \frac{1}{n+1}$ also DIV.

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eg 4. (Application of Limit comparison Test)

(5/6) Determine whether $\sum_{n=1}^{\infty} \frac{3n^2+n}{n^4+\sqrt{n}}$ is CONV or DIV. State the test you are using.

Solution: $a_n = \frac{3n^2+n}{n^4+\sqrt{n}}$, $b_n = \frac{3n^2}{n^4}$ (leading terms of a_n)

(Draw the conclusion for $\sum b_n$:) $b_n = \frac{3}{n^2}$ p-Series, $p=2>1$, $\sum b_n$ is convergent.

(Compute $\lim_{n\to\infty} \frac{a_n}{b_n}$)

$$\boxed{\lim_{n\to\infty} \frac{a_n}{b_n}} = \lim_{n\to\infty} \frac{3n^2+n}{n^4+\sqrt{n}} \cdot \frac{n^2}{3} = \lim_{n\to\infty} \frac{3n^4+\cancel{n^3}}{3n^4+\cancel{3\sqrt{n}}} = \lim_{n\to\infty} \frac{3n^4}{3n^4} \boxed{= 1}$$

Therefore, by the limit comparison test, since $\sum b_n$ is convergent, $\sum_{n=1}^{\infty} \frac{3n^2+n}{n^4+\sqrt{n}}$ is also CONV.

Remark: If the formula of a_n is such a ratio of two polynomials, then CHOOSE b_n via the $\boxed{\text{"leading term" rule}}$. Under this choice of b_n, $\lim_{n\to\infty} \frac{a_n}{b_n}$ will be 1, and $\sum a_n$ & $\sum b_n$ will be both CONV or DIV. Moreover, b_n (after simplification) is a p-series (up to some constant.)

More examples:

e.g.5. $\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n}$.

Solution 1: (Comparison Test). $a_n = \frac{\sqrt{2n+1}}{n}$, $b_n = \frac{1}{n}$. $a_n > b_n$. $\sum \frac{1}{n}$ DIV $\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n} \text{ DIV}}$.

Solution 2: (L.C. Test) $a_n = \frac{\sqrt{2n+1}}{n}$, $\boxed{b_n = \frac{\sqrt{2n}}{n} = \frac{\sqrt{2}}{\sqrt{n}}, \text{ p-Series}, p = \frac{1}{2} < 1, \text{ DIV}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{2n+1}}{n} \cdot \frac{\sqrt{n}}{\sqrt{2}} = \lim_{n\to\infty} \frac{\sqrt{2n^2+n}}{n\cdot\sqrt{2}} = \lim_{n\to\infty} \frac{\sqrt{2n^2}}{n\cdot\sqrt{2}} = 1$$

According to L.C.T., $\sum b_n = \sum \frac{\sqrt{2}}{\sqrt{n}}$ DIV implies that $\sum a_n = \sum \frac{\sqrt{2n+1}}{n}$ also DIV.

Compare with Geometric Series:

Remark: Exponential functions (of n) can also be viewed as "leading terms".

☆ e.g.6. For $a_n = \frac{1}{(7+3^n)^4}$, find a number x such that $\lim_{n\to\infty} \frac{a_n}{x^n} = L$, $0 < L < \infty$

(ww 4.(A)). And determine whether $\sum a_n$ is CONV or DIV? (via L.C. Test)

Hint: 'leading part' in a_n is $\frac{1}{(3^n)^4} = \frac{1}{3^{4n}} = \left(\frac{1}{3^4}\right)^n$

Let $x = \frac{1}{3^4}$, then $x^n = \left(\frac{1}{3^4}\right)^n$. $\lim_{n\to\infty} \frac{a_n}{x^n} = \lim_{n\to\infty} \frac{3^{4n}}{(7+3^n)^4} = \lim_{n\to\infty} \left(\frac{3^n}{7+3^n}\right)^4$

$\sum x^n = \sum \left(\frac{1}{3^4}\right)^n$ is CONV, $r = \frac{1}{3^4} < 1$ $\qquad = \lim_{n\to\infty} \left(\frac{1}{7\cdot 3^{-n}+1}\right)^4 = 1$

$\Rightarrow \sum a_n$ is CONV due to Limit C. Test.

More hints for WW.

ww 1.5, 1.6. $0 \le \sin^2(n) \le 1$ and $0 \le \arctan(n) \le \frac{\pi}{2}$ for $n = 1, 2, \ldots$

ww 2.3. $\sum_{n=1}^{\infty} \frac{(\ln n)^6}{n+5}$. One can consider $n \ge 2$ and $\ln n \ge \ln 2$ (some fixed positive constant)

ww 4. (B), (C), (D), when exponential functions and polynomials meet, choose
(☆☆) the exponential as "leading term".

ww 5.8. $\lim_{n\to\infty} \frac{(n+1)(6^2+1)^n}{6^{2n}} = \infty$, or $(n+1)\cdot\frac{(6^2+1)^n}{6^{2n}} > (n+1)\cdot\frac{6^{2n}}{6^{2n}} = n+1 > n$

ww 6.7. choose $b_n = \frac{1}{7n}$, compute $\lim_{n\to\infty} \frac{\tan\left(\frac{1}{7n}\right)}{\frac{1}{7n}} = 1$ and use limit C. Test.